Emission of correlated photon pairs from superluminal perturbations in dispersive media

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We develop a perturbative theory that describes a superluminal propagating in a dispersive medium and the subsequent excitation of the quantum vacuum zero-point fluctuations. We find a process similar to the anomalous Doppler effect: Photons are emitted in correlated pairs and mainly within a Cerenkov-like cone, one on the forward and the other in backward directions. The number of photon pairs emitted from the perturbation increases strongly with the degree of superluminality and under realizable experimental conditions; it can reach up to \( \sim 10^{-2} \) photons per pulse. Moreover, it is, in principle, possible to engineer the host medium so as to modify the effective group refractive index. In the presence of “fast-light” media, e.g., a material with group index smaller than unity, a further enhancement may be achieved and the photon emission spectrum is characterized by two sharp peaks that, in future experiments, would clearly identify the correlated emission of photon pairs.

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I. INTRODUCTION

Since the analysis of Schwinger [1] a great amount of research has been devoted to the study of particle production from the vacuum. In a seminal paper [2], Hawking predicted that the presence of a horizon around a black hole induces particle production. Actually, the existence of Hawking radiation does not require a gravitational collapse, but, rather, the key elements are a quantum field and an event horizon associated with a curved space-time metric [3–7]. Another mechanism providing particle production is the dynamical Casimir effect [8]. There, a two-dimensional quantum theory of a massless scalar field is considered that is influenced by the motion of a perfectly reflecting boundary (mirror). The vacuum expectation value of the energy-momentum tensor for an arbitrary mirror trajectory shows a nonvanishing radiation flux.

Here we consider a further effect by which particle production is induced by the superluminal motion of a perturbation of the refractive index in a dielectric medium with dispersion. Differing from the Hawking effect, the production of particles is not induced by the presence of a horizon but by the fast motion of the perturbation. Therefore, this effect can be better recognized in the context of the anomalous Doppler effect [9–12]. Following the description given by Ginzburg, light emitted by a generic moving source will be Doppler shifted according to the formula

\[
\omega' = \frac{\omega}{\gamma} \mid 1 - v/cn(\omega) \cos \theta \mid \]

where \( \omega' \) is the comoving reference frame value of the emitted frequency, \( \theta \) is the observation angle and \( \gamma = 1/\sqrt{1 - v^2/c^2} \). For \( v/cn(\omega) \cos \theta < 1 \) we have the normal Doppler effect but for \( v/cn(\omega) \cos \theta > 1 \), i.e., for a superluminal source we have the so-called anomalous Doppler effect (see Fig. 1). Emission of a photon on behalf of the moving source must, in general, correspond to a change in the internal state, e.g., a transition in energy from one level to another. In the normal Doppler regime, a positive-frequency photon, thus, will be emitted with a spontaneous transition from an upper to a lower energy state. In much the same way, in the anomalous Doppler regime a negative-frequency photon (positive frequency in the laboratory reference frame) will be spontaneously emitted in combination with a transition from a lower to a higher state. The energy required for this process is provided by the superluminal translational movement. Therefore, the superluminal source will continuously emit pairs of photons, one in the anomalous and the other in the normal Doppler regions [12]. Anomalous Doppler emission has never actually been observed before due to the obvious difficulty in actually realizing an experimental system in which to observe these effects.

In this work we extend the same concept proposed to generate event horizons in optical media: An intense laser pulse focused into the medium will induce a local increase in the refractive index and this create a perturbation in the dielectric medium (PDM) that travels locked with the laser pulse itself [5,7]. Laser pulses will typically travel at a well-defined group velocity so this kind of PDM is ideal for creating the analog of an event horizon. However, by properly choosing how the laser pulse overlaps with the dielectric medium, it is possible to generate a PDM that travels with any arbitrary velocity. For example, recent experimental results clearly show that it is possible to generate isolated peaks within the laser pulse that travel faster than the vacuum speed of light, \( c \) [13,14]. Within the context of these findings we, therefore, consider the problem of how a superluminal PDM interacts with the zero-point fluctuations of the surrounding quantum vacuum. This same problem was first tackled in Ref. [15], where it was shown that, in a dispersion-less medium, a superluminal PDM will spontaneously emit photons that are excited from the vacuum state. The photons appear as correlated pairs and their wavelength decreases from the far-infrared to the visible region as the PDM velocity is increased from \( \sim c \) to many times \( c \). Here we extend these results to a dispersive medium and we consider in more detail the nature of the correlated photons and possible experimental implementation. We also consider how the correlated photon pair emission may be enhanced by properly engineering the vacuum states, i.e., by using media in which the effective group velocity at the...
emitted frequencies is also superluminal, so-called fast-light media [16].

II. NONDISPERSE CASE

We start by briefly summarizing the main results and equations for the nondispersive case treated in Ref. [15].

An intense light pulse induces the perturbation of the refractive index through the optical nonlinear Kerr effect. This perturbation travels across the medium at the same speed of the light pulse. The medium was supposed to be nondispersive. The authors find that there is no pair production unless the pulse velocity \( v \) satisfies the condition \( v \geq \frac{c}{\sqrt{n_0}} \), where \( c \) is the velocity of light in vacuum and \( n_0 \) is the uniform and constant background refraction index. The model analyzed is based on the perturbative approach introduced by Schützhold et al. in Ref. [17] and the interaction representation for the electromagnetic field in the presence of a dielectric constant \( \epsilon(\tilde{x},t) \) depending on space and time is considered. Moreover, a uniform background and a constant value \( \epsilon_b \) for the dielectric constant are assumed. Thus, the disturbance is \( \epsilon(\tilde{x},t) - \epsilon_b \). Following Ref. [17], the interaction Hamiltonian is \( H_I = \xi \Pi^2 \) with \( \xi := \frac{1}{2} \frac{\epsilon_b}{\epsilon_b} - \frac{1}{\epsilon_b} \) and the number of particles created, labeled by the momentum \( \tilde{k} \) and the polarization \( \mu \), is

\[
N_{k,\mu} = \frac{\alpha \omega_0}{V} \int \frac{d^3k}{(2\pi)^3} \left[ 1 - \left( \frac{\epsilon_{\tilde{k},\mu}}{\epsilon_{\tilde{k},\mu}} \right)^2 \right],
\]

where \( \tilde{k} = (\tilde{k},\omega_0) = (k,ck/n_0) \), with \( k = |\tilde{k}| \), \( \epsilon_{\tilde{k}} = \tilde{k}/k \), the subscripts 1 and 2 label the particles of the emitted couple, and the particular shape of the perturbation is contained in the function \( \xi \). Considering a Gaussian behavior for the refraction index \( n^2 = \epsilon(x,t,y,z) \)

\[
n^2 = n_0^2 + 2n_0 \eta e^{-[(x-vt)^2+y^2+z^2]/2\sigma^2},
\]

the expression for the number of created particles becomes [18]

\[
N_{k,\mu} = 2^\alpha \sigma^2 \frac{\pi^2}{3} \frac{n^2}{n_0} \int d^3k \frac{\alpha_0 \omega_0}{V} e^{-\gamma |\tilde{k}| + |\tilde{k}|^2} \left[ \delta \left( \frac{\tilde{k}_1 + \tilde{k}_2}{2} \right) + \frac{c}{vn_0} (k_1 + k_2) \right] \left[ 1 - \left( \frac{\epsilon_{\tilde{k},\mu}}{\epsilon_{\tilde{k},\mu}} \right)^2 \right].
\]

The \( \delta \) function gives a constraint on the possible values of the momenta for the emitted particles.

III. DISPERSIVE CASE

A. Perturbation

When considering actual experiments, the previous model should only be considered as an approximation of a real medium and one has to take into account the presence of the dispersion. In this section we generalize the model of Ref. [15] for dispersive media. To introduce dispersion in the model we have to refine the approach leading to the expression for the number of emitted particles. The first step, up to Eq. (19), is to derive the Hamiltonian for a superluminal perturbation in a dispersive medium. This Hamiltonian will then be used, in a similar fashion to the treatment in Ref. [15], to derive the number of particles excited out of the vacuum state. Let us write the dispersive electromagnetic action as [19]

\[
S = \frac{1}{2} \int \tilde{E} \cdot \varepsilon \tilde{E} dx^4,
\]

where \( \varepsilon \tilde{E} \) is

\[
\varepsilon \tilde{E}(\tilde{r},t) = \varepsilon(\tilde{r},i\partial_t)\tilde{E}(\tilde{r},t).
\]

The variation of the action [Eq. (4)] with respect to the potential related to the electric field, \( \varepsilon \tilde{E} \), gives the Maxwell equations, with no free charges, for \( E \). We will not specify the operator \( \varepsilon \) now, but we will assume for it to be a symmetric operator. We will also assume that its inverse operator \( \varepsilon^{-1} \) exists. The unperturbed canonically conjugate momentum is then

\[
\Pi = \varepsilon \tilde{E},
\]

as can be computed from Eq. (4) using the Lagrangian formalism. If we expand in Fourier series the electric field and the conjugate momentum as

\[
\tilde{E}_\omega(\tilde{r}) = \int \frac{dt}{\omega} e^{-i\omega t} \tilde{E}(\tilde{r},t) dt,
\]

\[
\tilde{\Pi}_\omega(\tilde{r}) = \int \frac{dt}{\omega} e^{-i\omega t} \tilde{\Pi}(\tilde{r},t) dt,
\]

we can write

\[
\tilde{E} = \varepsilon^{-1} \tilde{\Pi} = \frac{1}{2\pi} \int \frac{d\omega}{\varepsilon(\tilde{r},\omega)} \tilde{\Pi}_\omega(\tilde{r}) d\omega.
\]

We introduce a perturbation \( \varepsilon(\tilde{r},i\partial_t) \rightarrow \varepsilon(\tilde{r},i\partial_t) + \delta \varepsilon(\tilde{r} - \tilde{v} t, i\partial_t) \). To proceed, we note that now the momentum is

\[
\tilde{\Pi} = \varepsilon \tilde{E} + \delta \varepsilon \tilde{E}.
\]

To describe the Hamiltonian we have to invert this operator in order to express the electric field in terms of the momentum. This can be done perturbatively. The perturbative series expansion is expressed in terms of a (small) parameter \( \eta \) so

\[
\delta \varepsilon(\tilde{r} - \tilde{v} t, i\partial_t) = \eta \varepsilon_1(\tilde{r} - \tilde{v} t, i\partial_t).
\]

We then write

\[
\tilde{E} = \tilde{E}_0 + \eta \tilde{E}_1 + \cdots,
\]

where the dots are higher orders in \( \eta \). We then have

\[
\tilde{\Pi} = \varepsilon \tilde{E}_0 + (\varepsilon_1 \tilde{E}_0 + \varepsilon \tilde{E}_1),
\]

where we have neglected the term \( \eta \delta \varepsilon \tilde{E}_1 = \eta^2 \varepsilon_1 E_1 \) being of second order in \( \eta \). Equation (13), compared with Eq. (6) at the
first order, gives
\[ \bar{\Pi} = \bar{\epsilon} \bar{E}_0, \tag{14} \]
\[ \epsilon \bar{E}_1 = -\bar{\epsilon}_1 \bar{E}_0. \tag{15} \]
Therefore, we may quantize the theory perturbatively by using the unperturbed momentum operator \( \bar{\Pi} \) and the perturbation
\[ \delta \bar{E} = \eta \bar{E}_1 = -\epsilon^{-1}(\delta \epsilon \epsilon^{-1} \bar{\Pi}). \tag{16} \]
We now calculate the integral of the whole space-time of the Hamiltonian density by taking the Legendre transformation of the action. Using the expression for the unperturbed momentum and of the perturbation, the Legendre transformation is
\[ \frac{1}{2} \int \bar{\Pi} \cdot \bar{E} dx^4 = \frac{1}{2} \int \bar{\Pi} \cdot \epsilon^{-1} \bar{\Pi} dx^4 \\
- \frac{1}{2} \int \bar{\Pi} \cdot \epsilon^{-1}(\delta \epsilon \epsilon^{-1} \bar{\Pi}) dx^4 + \cdots \\
= \frac{1}{2} \int \bar{\Pi} \cdot \epsilon^{-1} \bar{\Pi} dx^4 - \frac{1}{2} \int \epsilon^{-1} \bar{\Pi} \cdot \delta \epsilon \epsilon^{-1} \bar{\Pi} dx^4 + \cdots, \tag{17} \]
where the dots are higher-order terms and we have used the symmetry of \( \epsilon^{-1} \) (and dropped total time derivatives).
Thus, the Hamiltonian density is
\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \tag{18} \]
where \( \mathcal{H}_0 \) is the unperturbed Hamiltonian (with dispersion) and
\[ \mathcal{H}_I := -\frac{1}{2} \epsilon^{-1} \bar{\Pi} \cdot \delta \epsilon \epsilon^{-1} \bar{\Pi} \tag{19} \]
is the first-order perturbation, which reduces to the Hamiltonian used in Refs. [15,17] in the absence of dispersion.

The perturbative computation is then carried forward starting from this expression and, as in the nondispersive case, we adopt the interaction representation.

IV. THE MODEL

Consider a perturbation of the refractive index moving in the \( x \) direction with velocity \( v \), i.e., \( n(\omega) + \delta n(x - vt) \) and assume that the \( \omega \) dependence of the term \( \delta n \) is negligible. Thus, at the first order in \( \delta n \) the dielectric constant is \( \epsilon(\omega) = \epsilon(\omega)^2 + 2\epsilon(\omega)\delta n(x - vt) \). The perturbation term (19) may be taken as a model that we can write formally (from now on we will omit the subscript 0) as
\[ \mathcal{H}_I = -\frac{1}{2} \bar{E} \bar{E}_1 = -\bar{E} (n(i\partial_t)\delta n(x - vt)) \bar{E}, \tag{20} \]
where the apex \( S \) implies the Weyl prescription for the maximally symmetrized operator [20]. The perturbative approach used here may be justified by evaluating the main parameters under investigation. We expect to observe a similar behavior to the nondispersive case for which there is a peak of emission for \( \lambda_{\text{max}} = 3\sigma \) [15], where \( \sigma \) is the radius of the Gaussian-shaped perturbation. It is reasonable to expect that the perturbative approach works if the measured wavelength \( \lambda \) satisfies \( \sigma/\lambda >> \eta \). For example, if \( \eta \) is of the order of 0.01–0.001 the perturbative approach is justified for emitted wavelengths well below 100 nm if \( \sigma \sim 1 \mu\text{m} \). In this situation the operator appearing in Eq. (20) can be approximated by the more tractable one, 
\[ -\frac{1}{2} \bar{E} \bar{E}_1 = -\bar{E} (n(i\partial_t)\delta n(x - vt)) \bar{E} = -\bar{E} \delta n(x - vt) n(i\partial_t) \bar{E}, \]
in which the operator \( n(i\partial_t) \) acts just on the electric field \( \bar{E} \) and not on the variation of the refractive index \( \delta n(x - vt) \). This perturbation has the same form as the interaction Hamiltonian \( \mathcal{H}_I \) considered in Ref. [17], cf. Eq. (59). In the present case \( \delta n(x - vt) \) plays the role of \( \xi(t) \) and the electric field of \( \bar{E} \). These considerations allow us to apply the perturbative scheme of Ref. [17]. The expression of the electric field \( E \) is (see Ref. [21], Eq. (31), coherently adapted to the notation of Ref. [17]),
\[ \bar{E}(t) = i \sum_{K\mu}(2) \frac{1}{\omega_2(\omega)} \frac{1}{n(\omega)n_{\mu}(\omega)} \times [\bar{a}_{\ell\mu}(t)e^{i\epsilon} - a_{\ell\mu}(t)e^{-i\epsilon}] \bar{E}_{\ell\mu}, \tag{21} \]
where \( n_{\omega}(\omega) = \frac{\partial n(n_{\omega})}{\partial n_{\omega}} = n(\omega) + \omega \frac{\partial n(\omega)}{\partial \omega} = \frac{\epsilon - \bar{E}_{\ell\mu}}{\epsilon(\omega)} \) and \( k = (\bar{k},\omega) = (\bar{k},\epsilon,n(\omega)) \), with \( \bar{k} = |\bar{k}| \) and \( \bar{E}_{\ell\mu} = \bar{k}/k \). The action of the operator \( n(i\partial_t) \) on the electric field \( \bar{E} \) is
\[ n(i\partial_t)\bar{E} = n(i\partial_t) \int e^{-i\omega t} E_{\omega} dk \]
\[ = \int n(\omega) E_{\omega} dk, \tag{22} \]
where \( E_{\omega} \) are the Fourier modes in the expansion [Eq. (21)] of the electric field. The operator \( n(i\partial_t) \) acts on the integrand and can be expanded in power series in \( i\partial_t \). The temporal derivatives act just on the exponential and one obtains a power series in \( \omega \) that resums given \( n(\omega) \), see Ref. [19]. Thus, inserting Eq. (22) in the expression of the interaction Hamiltonian, the number of particles of momentum \( \bar{k} \) and polarization \( \mu_1 \) per unit of volume generated by the disturbance \( \mathcal{H}_I \) turns out to be
\[ N_{k\mu_1} = \langle \hat{N}_{k\mu_1} \rangle = \langle \psi(t \rightarrow -\infty) | \hat{N}_{k\mu_1} | \psi(t \rightarrow \infty) \rangle \]
\[ = \int d^4x_1 \int d^4x_2 |0 \rangle \langle 0 | \mathcal{H}_I(x_1) \hat{N}_{k\mu_1} \mathcal{H}_I(x_2) |0 \rangle, \tag{23} \]
where \( \hat{N}_{k\mu_1} = \bar{a}_{k\mu_1}^\dagger \bar{a}_{k\mu_1} \) is the number operator and \( \psi(t \rightarrow \pm\infty) \) are the free field, i.e., the solutions of the equation of motion when the interaction is switched off. Actually, we can suppose that the interaction is present just for a finite interval of time and far from this interval the dynamics is governed by the free Hamiltonian. To compute the vacuum expectation value in Eq. (23) we use Eq. (21), with the fields evolving freely, i.e., \( \bar{a}_{k\mu}(t) = \bar{a}_{k\mu} e^{-i\omega t} \), and, defining \( f(k) := \langle \frac{\epsilon - \bar{E}}{\epsilon(\omega)} \rangle^{1/2} \frac{1}{n(\omega)n_{\mu}(\omega)} \), we obtain the expression,
\[ N_{k\mu} = \int d^4x_1 d^4x_2 |0 \rangle \sum_{k_i,\mu_2} \sum_{k_2,\mu_3} \sum_{k_3,\mu_4} \sum_{k_4,\mu_5} \times \bar{a}_{k_1\mu_1}^\dagger \bar{E}_{k_2\mu_2}^\dagger \bar{E}_{k_3\mu_3}^\dagger \bar{E}_{k_4\mu_4}^\dagger \bar{E}_{k_5\mu_5} \]
\[ \times \delta n(x_1) \delta n(x_2) n(\omega_2) n(\omega_4) e^{-i(k_2+k_3+k_4+k_5) x_2} |0 \rangle, \tag{24} \]
where the argument $\chi$ of $\delta n$ is a four-vector and, in our particular case, it depends just on $x$ and $t$ as $x - vt$. We denote with a tilde the four-dimensional Fourier transform $\tilde{g}(\vec{k}, \omega) = \int dt dx dy dz \, g(x, t) e^{i \vec{k} \cdot \vec{x} - i \omega t}$ and, using the commutation rules for the operators $a_k$ and $a_k^\dagger$, $[a_{k_1,i_1}, a_{k_2,i_2}^\dagger] = \delta(k_1 - k_2)\delta_{i_1,j_2}$, we obtain,

$$N_{k,i_1} = f(k_1)^2 \sum_{k,i_2} \tilde{\delta}_n(k_1 + k_2)^2 f(k_2)^2 \delta(n(\omega_1) + n(\omega_2))^2 \times \left(\vec{e}_{k,i_1} \cdot \vec{e}_{k,i_2}\right)^2.$$  \hspace{1cm} (25)

The space-time dependence of the perturbation $\delta n$ is of the form $\delta n(t, x, y, z) \equiv \delta n(x - vt, y, z)$. Hence, using the variables $u = x - vt$, $w = x + vt$, the Fourier transform of $\delta n$ is

$$\tilde{\delta}_n(k_1 + k_2) = \frac{2\pi}{v} \left(k_{1x} + k_{2x} - \frac{c}{v} \left[\frac{k_1}{n(\omega_1)} + \frac{k_2}{n(\omega_2)}\right]\right) \times \int du dy dz \tilde{\delta}_n(u, y, z) e^{i(k_{1x} + k_{2x})u + i(k_1 + k_2)y + i(k_1 - k_2)z}. \hspace{1cm} (26)$$

### A. Superluminality emission cones

As for the nondispersive case, this last result is very meaningful from a physical point of view. The support of the $\delta$ distribution gives precise conditions for the possible states of the emitted photons. First, we analyze the condition given by the $\delta$ function in the nondispersive case. Its argument gives the constraint,

$$\left(k_{1x} - \frac{c}{vn_0}k_1\right) + \left(k_{2x} - \frac{c}{vn_0}k_2\right) = 0. \hspace{1cm} (27)$$

This equation implies that whenever $k_{1x}/k_1 > c/(vn_0)$, the second photon must satisfy $k_{2x}/k_2 < c/(vn_0)$, i.e., from every pair of emitted photons, one photon is emitted outside the Cerenkov cone, $\theta_0 = \arccos(c/(vn_0))$ (anomalous Doppler region, see Fig. 1), and the other is emitted outside the cone, in the normal Doppler region.

In the dispersive case we cannot always identify distinct cones of emission for the photons as before. The $\delta$ function gives the following condition:

$$\left(k_{1x} - \frac{c}{vn(\omega_1)}k_1\right) + \left(k_{2x} - \frac{c}{vn(\omega_2)}k_2\right) = 0. \hspace{1cm} (28)$$

Similarly to the nondispersive case, this equation implies that whenever $k_{1x}/k_1 > c/(vn(\omega_1))$, the second photon must satisfy $k_{2x}/k_2 < c/(vn(\omega_2))$. In terms of emission angles for the two photons, we find $\theta_1 < \arccos(\cos[\cos(\omega_1/vn(\omega_1))]$ and $\theta_2 > \arccos(\cos(\omega_2/vn(\omega_2)))$. Therefore, if $\arccos(\cos(\omega_1/vn(\omega_1))) > \arccos(\cos(\omega_2/vn(\omega_2)))$, the two cones overlap and, if $\arccos(\cos(\omega_1/vn(\omega_1))) < \arccos(\cos(\omega_2/vn(\omega_2)))$, there is a gap between them. In the first case, there is a region in which both photons can be emitted (differing from the nondispersive case), whereas in the second case there is a region in which no photon at all can be emitted. The two cases are shown in Figs. 2(a) and 2(b), respectively. The nondispersive case is obtained in the limiting situation in which $\arccos(c/(vn(\omega_1))) = \arccos(c/(vn(\omega_2)))$.

#### 1. Gaussian shape of the perturbation

We now focus attention on the problem of determining the actual number of emitted photons, starting from the assumption of a Gaussian shape for the refractive-index perturbation $\delta n = \eta e^{-\frac{1}{2}(k_1x + k_2x)^2 + k_1k_2}$. This form for the perturbation is what one may expect when a local refractive-index variation is induced by a laser pulse through the nonlinear Kerr effect. Indeed, laser pulses typically have a Gaussian-shaped intensity, $I$, profile along both longitudinal and transverse directions so $\delta n \sim I$, where $n_2$ is the nonlinear Kerr index, will have the same form. A typical value for $\eta$, for example, in fused silica glass, is $\sim 0.001$ when $n_2 \sim 3 \times 10^{-16}$ cm$^2$/W and we take $I \sim 3 \times 10^{12}$ W/cm$^2$. Inserting this $\delta n$ profile in Eq. (25), using $\sum_k \rightarrow V \int d^3\vec{k} \frac{1}{(2\pi)^3}$, and summing over the polarization states $\mu_1$ and $\mu_2$, we obtain,

$$N_{k_1} = \sigma^2 \eta^2 \frac{\omega_1}{v^2 V n^2(\omega_1)n^2_g(\omega_1)} \int d^3\vec{k} \, e^{-\frac{1}{2}k^2} \frac{\omega_2}{n^2(\omega_2)n^2_g(\omega_2)} \left[\delta(n(\omega_1) + n(\omega_2))^2 \times \left(\frac{k_1^2 + k_2^2}{k_1^2 + k_2^2}\right)^2 \right] \times \left\{\delta(k_{1x} + k_{2x}) - \frac{1}{\omega_1 + \omega_2}\right\}^2. \hspace{1cm} (29)$$

Note that in the nondispersive limit, i.e., $\frac{dn(\omega)}{d\omega} = 0$, the expression for the number of emitted particles $N_{k_1}$ reduces to Eq. (3).
2. Hyperbolic tangent shape of the perturbation

We can consider another shape for the perturbation of the refractive index. In particular, let us focus on a profile given by a hyperbolic tangent function. This allows us to investigate how the particular geometry of the perturbation affects the maximum of the wavelengths and the number of emitted particles. Moreover, a tanh shape may be adequately approximated by a very steep moving plasma front followed by a nearly constant plasma density that will decay (through electron recombination) over time scales of the order of ~100 fs in condensed media or ~1 ns in gaseous media [22]. This situation, therefore, may be adequately approximated by a tanh-like function. We also note that a plasma front will locally reduce the refractive index (δn < 0) as opposed to the Kerr effect that will usually increase the refractive index (δn > 0). Yet the result of the calculations depends not on the sign of δn but only on its amplitude, velocity, and physical dimensions.

The shape of the perturbation, therefore, is taken as

\[ δn = η \tanh \frac{x - vt}{\sigma_x} e^{-\frac{y^2}{2\sigma_y^2}} e^{-\frac{z^2}{2\sigma_z^2}}. \]

Again, we use the variables \( u = x - vt, w = x + vt, \) and \( y \) and \( z \) unchanged. Repeating the same computation as above, we obtain,

\[
N_k = \frac{\sigma_x^2 \sigma_y^2 \sigma_z^2 \eta^2 \pi^2}{v^2 V} \frac{ω_1}{2 n^2(ω_1) n_0^2(ω_1)} \int d^3k \frac{ω_2}{n^2(ω_2) n_0^2(ω_2)} \times \text{csch}^2 \left( \frac{πω_2}{2} (k_{1x} + k_{2x}) \right) e^{-σ^2 (k_{1y} + k_{2y})^2} e^{-σ^2 (k_{1z} + k_{2z})^2} \times \left[ n(ω_1) + n(ω_2) \right] \frac{k_{1x} k_{2x}}{k_{1x}^2} + \frac{(k_{1y} \cdot k_{2y})^2}{k_{1y}^2} \right] \times \left[ δ \left( k_{1x} + k_{2x} - \frac{1}{v} (ω_1 + ω_2) \right) \right]^2. \tag{30}
\]

Form these relations for the total emitted photon numbers it is immediately apparent that it is advantageous to keep a large a perturbation as possible with each of the longitudinal and transverse dimensions \( σ_x, σ_y, σ_z \) contributing equally in the multiplicative prefactor. As shown below, it turns out that the tanh profile emits roughly half the number of photons of a dimensionally similar Gaussian perturbation, indicating that the main role is played by the transient switch-on and -off of the perturbation. We also note the dependence on the group refractive index at the emission frequency. As we shall discuss below, this allows an additional degree of freedom for enhancing or controlling the photon emission process.

B. Numerical analysis: Gaussian shape

The nontrivial dispersion relation appearing in Eq. (29) makes the dispersive case more intricate to analyze, even numerically, than the nondispersive one. We perform the numerical analysis in fused silica where, for completeness, we show the full dispersion law given by the Sellmeier relation [23],

\[ n(λ) = \text{Re} \left[ 1 + \frac{a_1 λ^2}{λ^2 - l_1^2} + \frac{a_2 λ^2}{λ^2 - l_2^2} + \frac{a_3 λ^2}{λ^2 - l_3^2} \right]^{1/2}, \tag{31} \]

where

\[ a_1 = 0.473 115 591, \quad l_1 = 0.012 995 717 0, \]
\[ a_2 = 0.631 038 719, \quad l_2 = 4.128 092 20 \times 10^{-3}, \]
\[ a_3 = 0.906 040 498, \quad l_3 = 98.768 532 2. \]

As in the nondispersive case, the \( δ \) distribution in Eq. (29) gives a constraint that furnishes a relation between the momenta of the produced particles. In principle, a component of the momentum, say \( k_{2z}, \) could be expressed as a function of the other ones, so by integrating over \( k_{2z} \) and \( k_{2z} \) one could find explicitly the function \( N_{k_z} \) giving the distribution of the produced particles as a function of the momentum. However, for a generic dispersion law, the constraint cannot be analytically solved. We, therefore, study the critical points of the integrand of Eq. (29) that will call \( N_{k_1, k_2} \). This is a function of five variables, because the six components of the momenta are not all independent due to the constraint given by the \( δ \). The symmetry of the problem allows us to restrict the analysis to the planes \( k_{1z} = 0 \) and \( k_{2z} = 0 \), reducing the number of variables to three. This function describes the distribution of emitted couples as function of the momenta \( k_1 \) and \( k_2 \). Its maxima show the momenta of the couple of particles emitted with highest probability. Therefore, the effective number of emitted particles can be found integrating this distribution. We note that the maximum of the function \( N_{k_1, k_2} \) is located around \( θ_1 = 0 \) and \( θ_2 = π \). Indeed, in Eq. (29) all factors depend on \( k_{1x} \) and \( k_{2x} \) by their modulus except for the terms \( |k_{1} + k_{2}|^2 \) and \( (k_{1} \cdot k_{2})^2 \), depending on the relative orientation between the two wave vectors. For fixed \( k_1 \) and \( k_2 \), the function \( N_{k_1, k_2} \) is maximum for small \( |k_{1} + k_{2}|^2 \) and large \( (k_{1} \cdot k_{2})^2 \). Let \( k_{1} = (k_{1x}, k_{1y}, k_{1z}) = (k_{1x} \cos θ_1, k_{1x} \sin θ_1, 0) \) and \( k_{2} = (k_{2x}, k_{2y}, 0) = (k_{2x} \cos θ_2, k_{2x} \sin θ_2, 0) \), then

\[
|k_{1} + k_{2}|^2 = k_{1x}^2 + k_{2x}^2 + 2k_{1x}k_{2x} \cos(θ_2 - θ_1) \geq k_{1x}^2 + k_{2x}^2 - 2k_{1x}k_{2x},
\]

\[
(k \cdot k')^2 = |k_{1}k_{2}| \cos(θ_2 - θ_1) \leq k_{1x}^2 + k_{2x}^2.
\]

The two conditions are both satisfied for \( θ_2 - θ_1 = π, \) i.e., the two vectors point in opposite directions. The condition given by the \( δ \) function in Eq. (29) is \( k_{1x} + k_{2x} - \frac{ω_1 - ω_2}{v(ω_1)} = 0 \) that can be rewritten as \( k_{1x} \cos θ_1 - k_{2x} \cos θ_2 + \frac{ω_1 - ω_2}{v(ω_1)} = 0 \), where \( θ_1 \) is the angle between \( k_{1} \) and the \( x \) axis and, consequently, \( π + θ_2 \) the angle between \( k_{2} \) and the \( x \) axis. As observed before, the maximum of Eq. (29) depends only on \( k_1, k_2, \) and \( \mathbf{k} \cdot \mathbf{k}' \). Thus, at the maxima we have

\[
\frac{d}{dθ_1} \left[ N_{k_1, k_2}(k_{1x}, k_{2x}, \mathbf{k} \cdot \mathbf{k}') + λg(k_{1x}, k_{2x}, θ_1) \right] = \frac{d}{dθ_1} λg(k_{1x}, k_{2x}, θ_1) = λ(k_{1x} - k_{2x}) \sin θ_1 = 0,
\]

where \( λ \) is a Lagrange multiplier. With \( k_1 \) and \( k_2 \) in general being not equal and \( λ ≠ 0 \), we, therefore, have \( θ_1 = 0 \). We also
verified numerically that, indeed, the maximum of \( N_{k_1,k_2} \) is at \( \theta_1 = 0 \). We visualize this in Fig. 3(a), which shows an example \((\sigma = 2 \text{ and } \beta = 5)\) of the distribution of photon numbers in a logarithmic scale. The two straight lines correspond to the condition \( k_1^2 = k_2^2 \) and \( k_2^2 = k_2^2 \), and the curve represents the relation between \( \lambda_1 \) and \( \lambda_2 \) given by the \( \delta \) function in Eq. (29). As discussed, these three curves intersect at the maximum of \( N_{k_1,k_2} \), implying that, indeed, maximum emission occurs along the propagation direction \( \theta_1 = 0 \).

We perform the numerical analysis for perturbations with radius \( \sigma = 1 \, \text{\mu m} \) and \( \sigma = 2 \, \text{\mu m} \) and for increasing values of \( \beta \). As explained, to give an estimation of the number of emitted couples for every single PDM we have to evaluate the integrand of the Eq.(29), i.e., \( \mathcal{N}_{k_1,k_2} = \mathcal{N}/(d^2d^2d^2 \ell^d \ell^d) \). In the computation the value of \( \delta(0) \) has been approximated with \( L/(2\pi) \) [see Eq. (26)]. In the following we report all photon numbers in the function of wavelength rather than wave vector (thus indicated as \( N_{\lambda_1,\lambda_2} \)) and integrated over the azimuthal angles \( \phi_1 \) and \( \phi_2 \).

Figures 3(b) and 3(c) show the calculated photon numbers \( N_{\lambda_1,\lambda_2} \) in logarithmic scale for the value of \( \lambda_2 \) at which emission is maximum and for \( \eta = 0.001 \), two different values of the perturbation size, \( \sigma = 1 \) and \( 2 \, \text{\mu m} \), respectively. Figure 3(d) shows both wavelengths \( \lambda_1 = \lambda_{1\text{max}} \) and \( \lambda_2 = \lambda_{2\text{max}} \) at which emission is maximum and for \( 2 < \beta < 20 \). At lowest \( \beta_1 \), \( \beta_2 \) and \( \lambda_2 \) have the largest values and monotonically decreases and \( \beta \) increases. These points are also shown for the two values of \( \sigma \) considered, \( 1 \, \text{\mu m} \) (blue dots) and \( 2 \, \text{\mu m} \) (red dots). We observe that the photon emission peaks at wavelengths \( \lambda_{1\text{max}} \), which increases with the perturbation diameter. We also note that, as in the nondispersive case [15], \( \lambda_{1\text{max}} \) and \( \lambda_{2\text{max}} \) decrease from the midinfrared region into the visible region as \( \beta \), i.e., the perturbation velocity, increases. However, differing from the nondispersive case, we underline that, in general, \( \lambda_{1\text{max}} \) and \( \lambda_{2\text{max}} \) differ, i.e., the photons are correlated but at different wavelengths. However, for increasing \( \beta \) this difference becomes smaller with \( \lambda_{1\text{max}} \to \lambda_{2\text{max}} \) as \( \beta \to \infty \) (Fig. 3(d) and 4(b)).

Some numerical values of \( N_{\lambda_1,\lambda_2} \) from these graphs are given in Table 1[a] and (b). An estimation of the actual number of emitted particles can be obtained by integrating the function \( N_{\lambda_1,\lambda_2} \) over \( \lambda_1 \), \( \theta_1 \), and \( \theta_2 \). This furnishes, for \( \eta =

<table>
<thead>
<tr>
<th>\beta</th>
<th>\lambda_{1\text{max}}</th>
<th>\lambda_{2\text{max}}</th>
<th>N_{\lambda_1,\lambda_2}</th>
</tr>
</thead>
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<tr>
<td>9</td>
<td>0.75[−6]</td>
<td>0.87[−6]</td>
<td>1.74[−3]</td>
</tr>
<tr>
<td>10</td>
<td>0.68[−6]</td>
<td>0.78[−6]</td>
<td>2.91[−3]</td>
</tr>
</tbody>
</table>

TABLE I. Maxima of \( N_{\lambda_1,\lambda_2} \) for increasing \( \beta \) and two values of \( \sigma \). Numbers in square brackets denote powers of 10.
the standard dispersion law of the fused silica. (d) Log10
modified dispersion law of the fused silica. We emulate such superluminal group
emission at enhanced velocities by introducing a simple Lorentzian correction on
the produced couples. We investigate the correlation properties of
the number of emitted photons and, moreover, providing an
additional tool for investigating the correlation properties of
the (dispersion properties of the) vacuum states, increasing
the (dispersion properties of the) vacuum states, increasing
sion curve is strongly modified (without absorption). This
modified, e.g., into Bragg grating structures so the disper-
resonance close to the frequency of interest or is structurally
whereby the medium is chosen either to have an absorption
of light have been observed in several experiments [24–28],
emitted frequencies [see, e.g., Eq.(29)]. From our equations it
emission spectrum is strongly distorted but, most importantly,
two distinct maxima appear. These two peaks are related
to enhanced emission separately at
λ1 or at λ2. In virtue of the nature of the two emitted photons, enhancement at
one wavelength will necessarily lead to enhancement at the
correlated photon wavelength. Therefore, observation of these
two emission peaks in an experiment could be considered as
evidence of correlated photon emission from the superluminal
perturbation by measuring only in the forward direction. We
performed a similar analysis for the case of silicon, as this is
a widely used material in optics and waveguide technology
where both slow and fast light may be engineered. We obtain a
similar increase of the emitted particles as shown in Fig.6 for
a perturbation of dimension σ = 1 μm and velocity β = 4.

D. Fast light

We now return to a feature outlined earlier on, i.e., the
dependence of the photon emissivity on the group index at the
emitted frequencies [see, e.g., Eq. (29)]. From our equations it
is clear that by a reduction in the group velocity at ω1, and/or
ω2, may greatly influence the actual number of emitted photons
at both frequencies. Group velocities exceeding the speed of
light have been observed in several experiments [24–28],
whereby the medium is chosen either to have an absorption
resonance close to the frequency of interest or is structurally
modified, e.g., into Bragg grating structures so the disper-
sion curve is strongly modified (without absorption). This
property, therefore, can be exploited to effectively engineer
the (dispersion properties of the) vacuum states, increasing
the number of emitted photons and, moreover, providing an
additional tool for investigating the correlation properties of
the produced couples. We emulate such superluminal group
velocities by introducing a simple Lorentzian correction on
top of a background Sellmeier relation. We, therefore, obtain
a dispersion law with a narrow peak whose maximum slope
is placed in correspondence to the point of the highest photon
emission for a medium characterized only by the background
dispersion. We give an example of such a modified dispersion
law for the fused silica in Fig. 5(a) and the corresponding
group index, ng(ω1,λ2), in Fig. 5(b).

We then perform the analysis for a PDM of Gaussian shape
with η = 0.001, σ = 1 μm and traveling at β = 20 in such a
medium. In Fig. 5(c) we show the distribution of the emitted
couples in a medium with the background dispersion law and
in Fig. 5(d) the distribution produced by the same perturbation in
a medium with the Lorentzian correction to the dispersion. As a first comment, we observe that the number of emitted
particles increases by a factor of 10 due to the dependence of
Eq.(29) on the group indices n(ω1) and n(ω2). Moreover, the
emission spectrum is strongly distorted but, most importantly,
two distinct maxima appear. These two peaks are related to
enhanced emission separately at λ1 or at λ2. In virtue of the nature of the two emitted photons, enhancement at one wavelength will necessarily lead to enhancement at the correlated photon wavelength. Therefore, observation of these two emission peaks in an experiment could be considered as evidence of correlated photon emission from the superluminal perturbation by measuring only in the forward direction. We performed a similar analysis for the case of silicon, as this is a widely used material in optics and waveguide technology where both slow and fast light may be engineered. We obtain a similar increase of the emitted particles as shown in Fig. 6 for a perturbation of dimension σ = 1 μm and velocity β = 4.

V. CONCLUSIONS

Perturbative analysis of quantum fluctuation excitation from a traveling refractive-index perturbation indicates that emission of correlated photon pairs occurs only if v > c/n(ω). This effect bears a strong resemblance to the anomalous Doppler effect and, in this sense, represents the first detailed analysis of a setting in which the effect may actually be ob-
served. The number of emitted photon pairs is relatively small close to the threshold \( v = c/\eta(\omega) \) but increases significantly if \( v \) is much larger (e.g., 10–20\( \times \)) than \( c \). Our analysis fully accounts for material dispersion and, in doing so, introduces an intriguing dependence on the group index or group velocity of the emitted photons. This dependence and careful engineering of the host medium may be used to enhance photon emission by at least an order of magnitude and also provides direct evidence of correlated emission in the form of two correlated peaks in the output spectra.

From our analysis, it would seem that experiments are, indeed, feasible. Refractive-index perturbations of the order of those used in this work, \( \eta \sim 0.001 \), may be obtained by focusing laser pulses in fused silica or other nonlinear media and higher values have been observed. According to our results, this would result in a photon-pair emission rate of \( \sim 10^{-6} \) pairs/pulse at \( \beta = 20 \) in a 5-mm-long waveguide, i.e., \( \sim 1 \) photon per second if a MHz-repetition-rate laser is used. These numbers may be increased by an order of magnitude or more, e.g., by using fast-light media or higher PDM amplitudes. The high velocities used here for the perturbation may be obtained experimentally by sending an extended (approximately "plane wave") laser pulse at an angle onto the host medium that could be, for example, a waveguide (see Fig. 7). The pulse would intersect the waveguide with an angle \( \theta \) and the Kerr effect would create a PDM only at the intersection point that travels with speed \( c/\cos\theta \) (if we approximate the refractive index of air \( \sim 1 \)). Therefore, an incidence angle of \( \sim 80^\circ–85^\circ \) (i.e., close to normal incidence on the waveguide) would guarantee \( \beta \sim 5–20 \). We note that a similar approach has actually been used in previous pioneering experiments in which a dc field was converted to terahertz radiation by a superluminal laser pulse excitation of an array of biased capacitors [29]. In our case, we are exciting a superluminal PDM in a dielectric medium and correlated photon pairs would be generated and collected from the two ends of the waveguide.

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[18] Note that in Ref. [15] there was a misprint in the numerical constant beyond Eq. (3) of a factor 2\(^\times\).