Nonlinear Unbalanced Bessel Beams: Stationary Conical Waves Supported by Nonlinear Losses

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Nonlinear losses accompanying self-focusing substantially impact the dynamic balance of diffraction and nonlinearity, permitting the existence of localized and stationary solutions of the 2D + 1 nonlinear Schrödinger equation, which are stable against radial collapse. These are featured by linear, conical tails that continually refill the nonlinear, central spot. An experiment shows that the discovered solution behaves as a strong attractor for the self-focusing dynamics in Kerr media.

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One of the main goals of modern nonlinear wave physics is the achievement of wave localization, stationarity and stability. While in a one-dimensional geometry (e.g., in optical fibers), nonlinearity suitably balances linear wave dispersion, leading to the soliton regime, in the multidimensional case, nonlinearity drives waves either to collapse or instability. In self-focusing of optical beams, for instance, many stabilizing mechanisms, such as Kerr saturation, plasma-induced defocusing, or stimulated Raman scattering, have been explored, and are being the subject of intense debate, mainly in the context of light filamentation in air or condensed matter [1]. These mechanisms, however, are either intrinsically lossy, or due to the huge intensities involved, are accompanied by losses, which lead ultimately to the termination of any soliton regime. Similar pictures can be traced in all phenomena commonly discussed in the context of the nonlinear Schrödinger equation (NLSE), as Bose-Einstein condensates or Langmuir waves in plasma [2]. Nonlinear losses (NLL) arise in Bose-Einstein condensates from two- and three-body inelastic recombination, and as the natural mechanism for energy dissipation in Langmuir turbulence [3].

The question then arises of whether any stationary and localized (SL) wave propagation is possible in the presence of NLL. The response, as shown in this Letter, is affirmative. These SL waves cannot be ascribed to the class of solitary waves, but are instead *nonlinear conical waves* (as the nonlinear X waves [4]) of *dissipative* type, whose stationarity is sustained by a continuous refilling of the nonlinearly absorbed central spot with the energy supplied by linear, conical tails. These waves are not only robust against NLL, but find their stabilizing mechanism against perturbations in NLL themselves.

Among the linear conical waves [5], the simplest one is the monochromatic Bessel beam (BB) [6], made of a superposition of plane waves whose wave vectors are evenly distributed over the surface of a cone, resulting in a nondiffracting transversal Bessel profile. Despite the ideal nature of BBs (they carry infinite power), they not only have revealed to be a paradigm for understanding wave phenomena, but also have found applications as diverse as in frequency conversion, or in atom trapping and alignment [7]. Of particular interest for us is the finding [8] that the BB is describable in terms of the interference of two conical Hankel beams [5], carrying equal amounts of energy towards and outwards the beam axis, and yielding no net transversal energy flux in the BB.

What we demonstrate here is that a superposition of inward and outward Hankel beams with unequal amplitudes, i.e., an "unbalanced" Bessel beam (UBB), describes the only possible asymptotic form of the SL, nonsingular solutions of the 2D + 1 NLSE with NLL. For this reason, we call the solution "nonlinear UBB" (NL-UBB). We then show that NLL-UBB solutions do exist and are stable against radial perturbations, in the important case of Kerr nonlinearity with NLL. Unbalancing, that manifests as a reduced visibility of the radial Bessel oscillations, creates the required inward radial energy flux from the conical tails of the beam to refill the nonlinearly absorbed central spot, whose transversal pattern depends on the specific nonlinear phase (nondissipative) terms included in the NLSE. Contrary to linear conical waves, achievement of stationarity by refilling imposes generally a lower bound to the cone angle of the UBB.

In a self-focusing experiment in water, we demonstrate the self-generation of a NL-UBB from a Gaussian wave packet, which evidences that the NL-UBB acts as a strong attractor in the self-focusing dynamics, and hence a significant role of NLL in light filamentation.

To start with, we consider the 2D + 1NLSE

$$\partial_z A = \frac{i}{2k}\nabla_\perp^2 A + i\omega \frac{n_2}{c}|A|^2 A - \frac{\beta^{(K)}}{2}|A|^{2K-2}A, \qquad (1)$$

for the propagation of a light beam $E = A\exp(-i\omega t + ikz)$ of frequency ω in a Kerr medium (other nonlinear phase terms could be included as well) with NLL. In (1), $\nabla_\perp \equiv (\partial_x, \partial_y)$, $k = n\omega/c$ is the propagation constant (with n the refraction index and c the speed of light in vacuum), n_2 is the nonlinear refraction index, and $\beta^{(K)} > 0$ ($K = 2, 3, \ldots$) is the multiphoton absorption coefficient. For the real amplitude and phase [$A = a\exp(i\varphi)$, $a > 0$], (1) yields

$$\partial_z a^2 = -\frac{1}{k}\nabla_\perp \cdot (a^2 \nabla_\perp \varphi) - \beta^{(K)} a^{2K}, \qquad (2)$$

$$\partial_z \varphi = \frac{1}{2k}\left[\frac{\nabla_\perp^2 a}{a} - (\nabla_\perp \varphi)^2\right] + \frac{\omega n_2}{c} a^2. \qquad (3)$$

Stationarity of the intensity profile ($\partial_z a^2 = 0$) requires, from (2), $\varphi = \phi(x, y) + g(z)$. Then (3) imposes the linear dependence $g(z) = -\delta z$, where δ is a constant wave vector shift. Eqs. (2) and (3) then lead, for cylindrical beams, to the eigenvalue problem

$$a'' + \frac{a'}{r} + 2k\delta a - (\phi')^2 a + 2\frac{k^2 n_2}{n} a^3 = 0, \qquad (4)$$

$$-\frac{1}{k} 2\pi r \phi' a^2 = \beta^{(K)} 2\pi \int_0^r dr r a^{2K} \equiv N_r, \qquad (5)$$

[prime signs stand for d/dr, with $r \equiv (x^2 + y^2)^{1/2}$], with boundary conditions $a(0) \equiv a_0 > 0$, $a'(0) = 0$, $\phi'(0) = 0$, and the requirement that $a(r) \to 0$ as $r \to \infty$ for localization. Equation (5) establishes that in a SL beam, the power (per unit propagation length) entering into a disk of radius r must equal the power lost N_r within it. In absence of NLL ($\beta^{(K)} = 0$), this condition demands plane phase fronts ($\phi' = 0$) and no radial energy flux. This is the case of the sech-type Townes profile in Kerr media [9], associated to wave vector shift $\delta < 0$, and of weakly localized Bessel beams in linear [6] or Kerr media [10], with infinite power and wave vector shift $\delta > 0$.

With NLL, instead, the conditions of refilling (5) and localization [$a(r) \to 0$ as $r \to \infty$] require an inward radial power [lhs of (5)] that monotonically increases [rhs of (5)] with r up to reach, at infinity, a constant value equal to the total NLL, N_∞, assumed they are finite. Stationarity with NLL is thus supported by the continuous refilling of the more strongly absorbed inner part of the beam with the energy coming from its outer part. Phase fronts cannot be plane, since $\phi' \to -kN_\infty/2\pi r a^2$ as $r \to \infty$. As for the amplitude, the asymptotic value of ϕ' and the change $a(r) = b(r)/\sqrt{r}$ in (4), lead, when retaining only the slowest decaying contributions, to the Newton-like equation $b'' = -2k\delta b + k^2 N_\infty^2/4\pi^2 b^3$, that represents the "motion" of a particle in the potential $V(b) = k\delta b^2 + k^2 N_\infty^2/8\pi^2 b^2$. Since bounded trajectories $b(r)$ [leading then to $a \to 0$] under this potential can exist only for strictly positive δ, we conclude that *SL waves in media with NLL can only have positive wave vector shift,* $\delta > 0$. The solution of the Newton equation then yields the asymptotic behavior $a(r) = \{[c_1 + c_2\cos(2\sqrt{2k\delta}r + c_3)]/r\}^{1/2}$, with $c_1 > 0$, $c_1^2 - c_2^2 = kN_\infty^2/8\pi^2\delta$, that represents radial oscillations of contrast $C = |c_2|/c_1$ about an equilibrium point that approaches zero as $1/\sqrt{r}$. SL beams in media with NLL carry then infinite power, and have superluminal phase velocity ($\delta > 0$). These asymptotic features are more meaningfully expressed in terms of the UBB

$$A \simeq \frac{a_0}{2}[\alpha_{\rm out} H_0^{(1)}(\sqrt{2k\delta}r) + \alpha_{\rm in} H_0^{(2)}(\sqrt{2k\delta}r)]e^{-i\delta z}, \qquad (6)$$

formed by two nondiffracting Hankel beams of the first and second kind [5], of same cone angle $\theta = \sqrt{2\delta/k}$, but different weights, $\alpha_{\rm out}$ and $\alpha_{\rm in}$, that must be related by

$$a_0^2(|\alpha_{\rm in}|^2 - |\alpha_{\rm out}|^2)/k = N_\infty. \qquad (7)$$

The balanced superposition of Hankel beams (e.g., $\alpha_{\rm out} = \alpha_{\rm in} = 1$) just gives the original, nondiffracting Bessel beam $a_0 J_0(\sqrt{2k\delta}r)\exp(-i\delta z)$, with no net radial energy flux, and with oscillations of maximum contrast $C = 1$. Instead, unbalancing creates an inward radial power that is manifested in a lowering of the contrast $C = |c_2|/c_1 = 2|\alpha_{\rm in}||\alpha_{\rm out}|/(|\alpha_{\rm in}|^2 + |\alpha_{\rm out}|^2)$ of the Bessel oscillations, reaching $C = 0$ (no oscillations) in a pure Hankel beam.

We stress that this analysis holds irrespective of the nonlinear phase terms in the NLSE (Kerr nonlinearity, Kerr saturation, ...), since they rely on the only assumption of finite NLL. We can therefore state that *the conical UBB represents the only possible asymptotic form of SL waves in nonlinear media when the effects of NLL are taken into consideration.* Note that the case of linear losses is excluded, since $N_\infty = \infty$ for $K = 1$. It should be also clear that the actual existence of a SL solution of the NLSE, and the characteristics of its linear asymptotic UBB (cone angle, $\alpha_{\rm in}$ and $\alpha_{\rm out}$), depends on the particular nonlinear phase terms in the NLSE.

We first solved numerically (4) and (5) without the Kerr term, to appreciate the effects of pure NLL [Fig. 1(a)], and found that SL solutions [$a(r) \to 0$] exist indeed with any peak intensity $I_0 = a_0^2$ and wave vector shifts

$$\delta > g_K \beta^{(K)} I_0^{K-1}, \qquad (8)$$

where $g_K = 1.67, 0.27, 0.19, 0.16...$ for $K = 2, 3 ...$ As seen in Fig. 1(a), shortly away from the central peak, the radial profile becomes undistinguishable from that of the UBB of same δ, matched NLL N_∞ and contrast C. Moreover, *pure NLL creates a gap in the allowed UBB cone angles* $\theta = \sqrt{2\delta/k}$ *that increases with intensity* [Fig. 1(b)], and that becomes significant at intensities comparable to the characteristic intensity $(k/\beta^{(K)})^{1/(K-1)}$. Intuitively, this gap originates from the fact that the SL profiles widen and delocalize [Fig. 1(a)] as the cone angle diminishes. This causes the total NLL, N_∞, to increase, a situation that cannot be sustained down to the limit of zero cone angle ($\delta \to 0$), which would not allow for any radial energy flux.

The extension to include nonlinear phase terms in the NLSE can be readily understood from the case of pure NLL. Figure 1(b) shows the permitted cones angles in the case of focusing Kerr nonlinearity ($n_2 > 0$), obtained from numerical integration of (4) and (5) with $K = 4$ (similar results hold for other values of K). The modification of the allowed region of cone angles can be attributed to the nonlinear phase shift at the central spot. In fact, assuming that the existence of a localized solution is now determined by the effective wave vector shift $\delta_{\rm eff} = \delta + \delta_{\rm nl}$, with $\delta_{\rm nl} = kn_2 I_0/n$ for Kerr nonlinearity, we replace δ with $\delta_{\rm eff}$ in Eq. (8), to obtain

$$\delta > \max\{g_K \beta^{(K)} I_0^{K-1} - kn_2 I_0/n, 0\} \tag{9}$$

(where we set to zero negative values) as an accurate expression for the allowed linear wave vector shifts in Kerr media [see Fig. 1(b) for the associated cone angles $\theta = \sqrt{2\delta/k}$]. Similar expressions as (9) can be obtained for other nonlinear phase effects.

When NLL dominate over Kerr nonlinearity [right part of Fig. 1(b), or $g_K \beta^{(K)} I_0^{K-1} \gg kn_2 I_0/n_2$] the SL profiles (not shown) do not substantially differ from the case of pure NLL. For the Kerr-dominated case [left part of Fig. 1(b), or $g_K \beta^{(K)} I_0^{K-1} \ll kn_2 I_0/n_2$], Fig. 1(c) shows a representative SL profile. The central peak and inner rings are nearly identical to the Kerr-compressed, Bessel-like beam in lossless Kerr media [10], as seen in Fig. 1(c), though the small NLL leads to a slight contrast reduction. The central peak can be approached by the Bessel beam $a_0 J_0(\sqrt{2k\delta_{\rm eff}} r)$ [10], and hence its width by $1/\sqrt{2k\delta_{\rm eff}}$. Outer rings gradually shrink up to become phased with the asymptotic UBB of wave vector shift δ [Fig. 1(c)]. Note that, as opposed to the pure NLL regime, the beam does not widen indefinitely as the cone angle diminishes down to its lower bound ($\theta \to 0^+$ in the Kerr-dominated case), but is limited to a maximum beam width $1/\sqrt{2k\delta_{\rm eff}} = \sqrt{n/2k^2 n_2 I_0}$. This fact entails, contrary to the pure NLL regime, a limitation to N_∞ as $\theta \to 0^+$, explaining why the unbalance mechanism for replenishment can support stationarity at arbitrarily small, but positive cone angles.

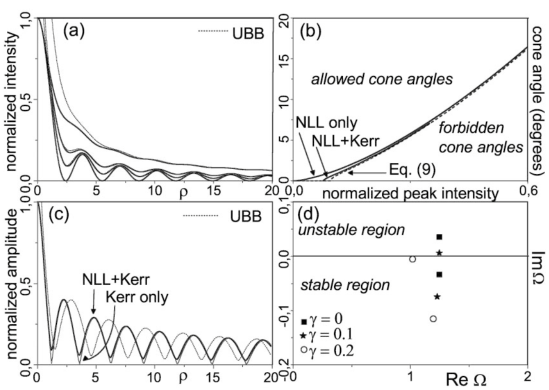

FIG. 1. (a) For pure NLL ($K = 4$), radial profiles a^2/a_0^2 of SL beams with decreasing $\delta/\beta^{(K)} I_0^{K-1} = \infty, 0.33, 0.25, 0.20$ (from lower to higher ones), and their asymptotic UBB. Normalized radial coordinate is $\rho = \sqrt{2k\delta} r$ (b) Allowed cone angles $\theta = \sqrt{2\delta/k}$ versus peak intensity I_0 normalized to the intensity $(k/\beta^{(K)})^{1/(K-1)}$ in the cases of pure NLL, and NLL + Kerr [numerically calculated and from Eq. (9)]. (c) For NLL+Kerr, amplitude profile a/a_0 for $\delta/\beta^{(K)} I_0^{K-1} = 0.24$ and $kn_2 I_0/n\delta = 6.07$, and its asymptotic UBB. The pure Kerr case is also shown. (d) Eigenvalue spectrum of the perturbations with $m = 0$ to the SL solutions with Kerr nonlinearity $g = 1$ and NLL ($K = 4$) $\gamma = 0, 0.1, 0.2$.

We also studied the stability of the NL-UBB solutions against perturbations. With the dimensionless quantities $\rho = \sqrt{2k\delta} r$, $\xi = \delta z$ and $\mathcal{A} = A/a_0$, we rewrite (1) as

$$\partial_\xi \mathcal{A} = i\nabla_\rho^2 \mathcal{A} + ig|\mathcal{A}|^2 \mathcal{A} - \gamma |\mathcal{A}|^{2K-2} \mathcal{A} \tag{10}$$

where $\nabla_\rho^2 = \partial_\rho^2 + (1/\rho)\partial_\rho$, $g = \omega n_2 a_0^2/c\delta$ and $\gamma = \beta^{(K)} a_0^{2K-2}/2\delta$. Following a standard Bogoliubov-deGennes procedure [11], we introduce a perturbed solution

$$\mathcal{A} = \mathcal{A}_0(\rho, \xi) + [u(\rho) e^{-i\Omega\xi + im\theta} - v^*(\rho) e^{i\Omega^*\xi - im\theta}] e^{-i\xi}, \tag{11}$$

where $\mathcal{A}_0(\rho, \xi) = a(\rho)\exp[i\phi(\rho)]\exp(-i\xi)$ is a SL solution of (10), θ is the polar angle, and $m = 0, 1 \ldots$, into (10), to obtain, upon linearization, the (non-self-adjoint) eigenvalue problem

$$\begin{aligned} \Omega u &= Hu - iK\gamma a^{2K-2} u + g a^2 e^{2i\phi} v \\ &\quad + i(K-1)\gamma a^{2K-2} e^{2i\phi} v \\ -\Omega v &= Hv + iK\gamma a^{2K-2} v + g a^2 e^{-2i\phi} u \\ &\quad - i(K-1)\gamma a^{2K-2} e^{-2i\phi} u, \end{aligned} \tag{12}$$

where $H = -\nabla_\rho^2 + m^2/\rho^2 - 1 - 2ga^2$, with boundary conditions $u(\rho) \to 0$, $v(\rho) \to 0$ as $\rho \to \infty$. The existence of an eigenvalue Ω with $\mathrm{Im}\Omega > 0$ would render unstable the SL solution $\mathcal{A}_0$. To obtain numerically the eigenvalues, we transform (12) into an algebraical eigenvalue problem by introducing a mesh of small size s in the range $[\rho = 0, \rho = L]$ with large L, writing the differential operators as finite differences, and imposing the boundary conditions $u(L) = v(L) = 0$. The numerical diagonalization of the $2N \times 2N$ matrix (N being the number of mesh points) of the eigenvalue problem provides a set of $2N$ eigenvalues. The behavior of the continuous distribution of eigenvalues of (12) is inferred by extrapolating the results to the limit $s \to 0$ and $L \to \infty$. We investigated increasing N up to 4000 ($L = 400$, $s = 0.1$), as limited by the accessible memory of our computational facility.

Figure 1(d) shows a typical eigenvalue spectrum for radial perturbations ($m = 0$) to three SL solutions with same Kerr nonlinearity ($g = 1$) and increasing NLL. Eigenvalues with $\mathrm{Im}\Omega = 0$ are not shown for clarity. Clearly, *the effect of NLL is to decrease the imaginary part of the eigenvalues, driving the system towards stability*: The Bessel-like solutions in pure Kerr media (squares) are unstable. At NLL strength $\gamma = 0.1$ (stars), the positive imaginary parts are strongly reduced, and at $\gamma = 0.2$ (circles) no signs of instability are present. We

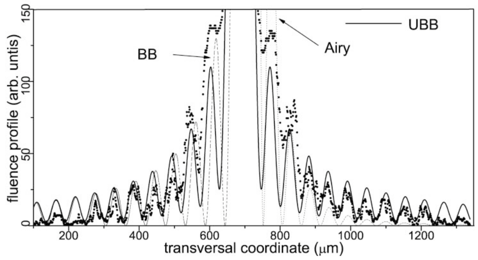

FIG. 2. Measured fluence profile (dots), fitted UBB, and half Bessel (left) and Airy (right) profiles.

performed the same analysis for $m = 1, 2, \ldots$ For dipolar perturbations ($m = 1$) no instability emerges neither in the pure Kerr nor in the Kerr + NLL model. In contrast, both systems turned out to be unstable for quadrupolar and higher-order perturbations, a result that can be related to the fact these perturbations involve modulation far from the central spot, where intensity is weak and so NLL cannot play its stabilizing role.

The experiment that we present demonstrates that the NL-UBB stationary solution indeed acts as strong attractor for the transient dynamics of light beam self-focusing in (weakly dispersive) Kerr media. Recently [12] we have shown that light filaments in water do not behave as solitonlike beams, but they spread after being clipped by an aperture and reconstruct themselves after being blocked by a stopper, as expected for genuine conical waves [13]. Here we concentrate our attention in the beam-periphery structure in order to demonstrate that filaments are indeed conical and, more precisely, NL-UBB waves. To this end we modified the diagnostic by adopting professional digital photo camera (Canon-EOS D30), which has the unique advantage of permitting strong local saturation (by the central spike) without any blooming effect. The experiment was done by launching a spatially filtered, collimated, 200 fs, 0.1 mm, 1.5 μJ, 527 nm Gaussian wave packet into a 31 mm water-filled cuvette. By using an imaging spectrograph we verified that, at the specified pump energy, no relevant spectral broadening occurs, the self-phase modulation occurring in the sole spatial domain. Figure 2 shows the measured fluence profile at the output facet of the nonlinear sample, and a fitted UBB profile. BB and Airy profiles (only one half, for clarity) are also shown for comparison. One can appreciate: (i) The Bessel-like decay ($\sim 1/r$) of the measured profile in a fairly vast region of the beam, which distinguishes it sharply from any (Airy-type) aperture-diffraction pattern (with faster decay $\sim 1/r^2$). (ii) The accurate fitting of the UBB radial modulations to those of the measured profile in the full recorded area, which even reproduces the increasing frequency of the modulations (respect to a BB) towards the beam center, attributable to Kerr self-focusing. (iii) The reduction in modulation contrast (compared to a BB), which is a signature of unbalance between inward and outward conical power flows.

In conclusion, we have reported on the existence, characteristics, stability and experimental relevance of nonlinear unbalanced-Bessel-beams (NL-UBB), i.e., the stationary and localized solutions of the 2D + 1 NLSE in the presence of NLL. NL-UBB are asymptotically linear conical waves, carrying a net inward power flux that compensate for the NLL. We have shown that the UBB asymptotics is the sole compatible with NLL, no matter which are the specific nonlinear phase terms in the NLSE. Therefore NL-UBB are possible solutions also of nonintegrable NLSEs as well as of the Gross-Pitaevskii equation for Bose-Einstein condensates. Owing to the spatiotemporal analogy, the results directly apply to pulse propagation in planar wave guides with anomalous dispersion. More generally, similar waves should exist also in the case of 3D + 1NLSE with NLL, describing full spatiotemporal localization in bulk media with NLL. The unique property of long-range stationarity in the presence of energy transfer to matter (or to other waves) makes NL-UBB ideal for several applications including deep-field nonlinear microscopy, laser micro machining, laser writing of channel wave guides, charged-particle acceleration, creation of long and stable plasma channels in atmosphere and, of course, energy transfer between different types of waves (e.g., optical $\rightarrow X$, or optical $\rightarrow$ teraHertz).

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